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Riccati map on $L_0^2(\mathbb{T})$ and its applications

Thomas Kappeler [*,1], Peter Topalov [1]

Institute of Mathematics, University of Zürich, Zürich, Switzerland

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Abstract

This paper is concerned with the spectral properties of the Schrödinger operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with periodic potential q from the Sobolev space $H^{-1}(\mathbb{T})$. We obtain asymptotic formulas and a priori estimates for the periodic and Dirichlet eigenvalues which generalize known results for the case of potentials $q \in L^2(\mathbb{T})$. The key idea is to reduce the problem to a known one—the spectrum of the impedance operator—via a nonlinear analytic isomorphism between $L_0^2(\mathbb{T})$ and the Sobolev space $H_0^{-1}(\mathbb{T})$.

Keywords: Hill operator; Riccati transform

1. Introduction

This paper is devoted to the spectral properties of the Schrödinger operator

$$L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$$

* Corresponding author.
E-mail address: tk@math.unizh.ch (T. Kappeler).
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with "singular" potential q from the Sobolev space $H^{-1}(\mathbb{T})$ viewed as an unbounded operator on $H^{-1}(\mathbb{T})$ where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Our approach is based on a nonlinear representation—clearly of independent interest—of the elements of the Sobolev space

$$H_0^{-1}(\mathbb{T}) \stackrel{\text{def}}{=} \{g \in H^{-1}(\mathbb{T}) \mid \langle q, 1\rangle = 0\}$$

in terms of a unique function from

$$L_0^2(\mathbb{T}) \stackrel{\text{def}}{=} \left\{q \in L^2(\mathbb{T}) \;\middle|\; \int_{\mathbb{T}} q(x)\,dx = 0\right\},$$

referred to as Riccati representation. Here $\langle\cdot,\cdot\rangle$ denotes the natural dual pairing $H^{-1}(\mathbb{T}) \times H^1(\mathbb{T}) \to \mathbb{R}$ between distributions and test functions.

Let $\rho \in H^1(\mathbb{T})$ with $\min_{x\in\mathbb{T}} \rho(x) > 0$ and define $r \stackrel{\text{def}}{=} \rho'/\rho$. By the transformation $y \stackrel{\text{def}}{=} \rho\tilde{y}$, the equation $-\tilde{y}'' - 2r\tilde{y}' = \lambda\tilde{y}$ becomes the Schrödinger equation $-y'' + qy = \lambda y$ with $q = r' + r^2$. It was suggested in [7] that the spectral properties of the Schrödinger operator L_q could be deduced from the spectral properties of the impedance operator

$$T_r(u) \stackrel{\text{def}}{=} -\big(\rho^2 u'\big)'/\rho^2,$$

whose spectral theory is well-developed [4,5,15,16,19,22]. In the present paper we prove that for any $\alpha \geqslant 0$ the *Riccati map* $r \mapsto r' + r^2 - \|r\|^2$ maps $H_0^\alpha(\mathbb{T})$ onto $H_0^{\alpha-1}(\mathbb{T})$ where $\|r\|^2 = \int_0^1 r^2(x)\,dx$ and

$$H_0^\alpha(\mathbb{T}) \stackrel{\text{def}}{=} \left\{f \in H^\alpha(\mathbb{T}) \;\middle|\; \int_0^1 f\,dx = 0\right\}.$$

In fact it is a real-analytic isomorphism between $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ (see Section 3). Our proof is simple and based on elementary spectral properties of the Schrödinger operator. In particular, the properties of the first eigenvalue $\lambda_0(q)$ of the Schrödinger operator and the corresponding normalized eigenfunction $f_0(q)$ are essential for our approach.

The main result of this paper is summarized in the following theorem.

Theorem 1. *The Riccati map $R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T})$ is a real-analytic isomorphism. For any $r \in L_0^2(\mathbb{T})$, the impedance operator T_r and the associated Schrödinger operator L_q with $q = R(r)$ have, up to a translation by $\|r\|^2$, the same periodic and Dirichlet spectrum. Moreover, $R^{-1}(q)$ is given by*

$$R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q),$$

where $f_0(\cdot, q)$ is an eigenfunction (in $H^1(\mathbb{T})$) corresponding to the first (lowest) periodic eigenvalue $\lambda_0(q)$ of L_q which can be proved to vanish nowhere.

Our analysis of the Schrödinger operator L_q for potentials in $H^{-1}(\mathbb{T})$ is mainly motivated by its use for studying the well-posedness of the KdV equation in $H^{-1}(\mathbb{T})$. To this end we present in Section 4 several applications which are used in a crucial way to extend the construction of action angle variables for KdV from $L^2(\mathbb{T})$ (cf. [9]) to $H^{-1}(\mathbb{T})$ (cf. [8]) and obtain the following theorem (cf. [12]).

Theorem 2. *KdV is globally C^0-well-posed in $H^\beta(\mathbb{T})$ for any $-1 \leqslant \beta < -1/2$. In particular, the solution map $\mathcal{S}: H^\beta(\mathbb{T}) \to C(\mathbb{R}, H^\beta(\mathbb{T}))$ has the group property, $\mathcal{S}(t+s, q) = \mathcal{S}(t, \mathcal{S}(s, q))$ $(\forall t, s \in \mathbb{R},\ q \in H^\beta(\mathbb{T}))$ and for any $t \in \mathbb{R}$, the flow $\mathcal{S}_t: H^\beta(\mathbb{T}) \to H^\beta(\mathbb{T})$, $q \mapsto \mathcal{S}(t, q)$, is a homeomorphism.*

We remark that the well-posedness results for KdV on the circle known so far required initial data in $H^\beta(\mathbb{T})$ with $\beta \geqslant -1/2$ [2,6,14].

How the paper is organized

In Section 2 we prove auxiliary results needed for the proof of Theorem 1 given in Section 3. Section 4 is devoted to applications of the Riccati representation mentioned above: first we prove spectral results for the Hill operator L_q using results of the impedance operator (cf. [15]). Using results on the impedance operator of [4,5], we then give results on the Dirichlet spectrum of L_q which generalize results of [31] to the case of singular potentials $q \in H^{-1}(\mathbb{T})$. Finally, we extend the notion of discriminant to potentials in $H^{-1}(\mathbb{T})$ and prove the isospectral invariance of the Riccati map. To make the paper self-contained we include three appendices. For the convenience of the reader we present in Appendices A and B known results on the spectrum of T_r respectively L_q used in this paper.

Related work

Closely related to the Riccati map is the Miura map $r \mapsto r' + r^2$ which Miura introduced in [27,28] in the search of integrals of motion for the Korteweg–de Vries equation. In fact, this transform maps smooth solutions of the modified Korteweg–de Vries equation to solutions of the Korteweg–de Vries equation. The Miura map also turned up in the study of elliptic boundary value problems. Motivated by work of Ambrosetti and Prodi [1], McKean and Scovel [26] studied, among other nonlinear maps, the Miura map and its geometry as a map from $H^1(\mathbb{T})$ to $L^2(\mathbb{T})$, exhibiting a global fold structure. See also [3] for further results in this direction.

The Riccati map is the composition of the restriction of the Miura map to the inverse image of the fold, with a translation by $-\|r\|^2$. To study the spectral properties of the Schrödinger operator L_q for singular potentials $q \in H^{-1}(\mathbb{T})$, Korotyaev considered in [17] an integrated version of the Riccati map and proved—as part of an investigation of a whole class of nonlinear maps—that the Riccati map is real-analytic and injective [17, Theorem 2.5]. Independently of our work, Korotyaev has shown in [18] by a different, more complicated proof that the Riccati map is surjective with real-analytic inverse. Actually, in [18], Korotyaev proves more than this result, solving an inverse problem for the Schrödinger operator with singular periodic potentials $q \in H^{-1}(\mathbb{T})$. As already mentioned above, our interest in the Riccati map comes from its use for the construction of action-angle variables for KdV in $H^{-1}(\mathbb{T})$ (cf. [8]) which allowed us to improve on existing results of the initial value problem of KdV—see Theorem 2.

The spectral theory of the operators L_q with potentials $q \in L^2(\mathbb{T})$ has been extensively studied—see, e.g., [21,24,25]. For potentials q which are distributions, $q \in H^\alpha(\mathbb{T})$, $-1 \leqslant \alpha < 0$, the spectral theory of L_q was partially developed in [7,29,30,32,33], using a direct approach instead of the spectral theory of the impedance operator T_r. However, the results obtained do not suffice for our construction of action-angle variables for KdV in $H^{-1}(\mathbb{T})$ [8].

This paper is based on the preprint [10] (see [11] for an abridged version).

Notations

The following notations are used throughout the paper. Denote by $h^{\beta,n}$ ($\beta \in \mathbb{R}, n \in \mathbb{Z}$) the Hilbert space of sequences $\{x_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ with finite norm

$$\|x\|_{\beta,n} \stackrel{\text{def}}{=} \left(\sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} |x_k|^2 \right)^{1/2},$$

where $\langle s\rangle \stackrel{\text{def}}{=} |s|+1$. The scalar product in $h^{\beta,n}$ is given by

$$(x,y)_{\beta,n} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} x_k \bar{y}_k.$$

By definition, $h^\beta \stackrel{\text{def}}{=} h^{\beta,0}$. By $\mathbb{T}_l$, $l>0$, we denote the one-dimensional torus $\mathbb{T}_l \stackrel{\text{def}}{=} \mathbb{R}/l\mathbb{Z}$. The Sobolev spaces $H^m(\mathbb{T}_l)$ and $H^m[0,1]$, $m \in \mathbb{N}$, are defined by

$$H^m(\Omega) \stackrel{\text{def}}{=} \bigl\{ f : \Omega \to \mathbb{R} \bigm| f^{(k)} \in L^2(\Omega),\ k = 0,1,\dots,m \bigr\},$$

where $f^{(k)}$ is the kth distributional derivative of the function f and Ω denotes the torus $\mathbb{T}_l$ or the interval $[0,1]$, respectively. The scalar product in $H^m(\Omega)$ is defined by

$$(f,g)_m \stackrel{\text{def}}{=} \sum_{k=0}^{m} \bigl(f^{(k)}, g^{(k)}\bigr),$$

where $(\cdot,\cdot)$ denotes the L^2-scalar product. For real $\alpha \geqslant 0$ the Sobolev spaces $H^\alpha(\mathbb{T}_l)$ and $H^\alpha[0,1]$ can be defined in a standard manner, for example, by interpolation (see [23, Chapter 1]). By definition, $H^{-\alpha}(\Omega)$ is the dual space of $H^\alpha_c(\Omega)$, i.e., $H^{-\alpha}(\Omega) \stackrel{\text{def}}{=} (H^\alpha_c(\Omega))'$ where $H^\alpha_c(\Omega)$ is the closure in $H^\alpha(\Omega)$ of the space $C^\infty_c(\Omega)$ of smooth functions with compact support in the interior of Ω. The norm of f in $H^\alpha(\Omega)$ is denoted by $\|f\|_\alpha$ and for $\alpha = 0$ we write $\|f\| = \|f\|_0$. The distributional derivative

$$\frac{d}{dx} : H^m(\Omega) \to H^{m-1}(\Omega), \quad m \in \mathbb{N},$$

can be extended in a natural way to a bounded operator

$$\frac{d}{dx} : H^\alpha(\Omega) \to H^{\alpha-1}(\Omega)$$

for arbitrary $\alpha \in \mathbb{R}$. By $H^\alpha_0(\mathbb{T}_l)$ we denote the linear subspace of elements $f \in H^\alpha(\mathbb{T}_l)$ with mean value zero,

$$[f] \stackrel{\text{def}}{=} \int_{\mathbb{T}_l} f\,dx = 0.$$

Note that the Sobolev spaces $H^\alpha(\mathbb{T}_l)$, $\alpha \in \mathbb{R}$, can be identified (up to an equivalence of the norms) with the space of Fourier series $\sum_{k\in\mathbb{Z}} \hat{f}_k e^{2i\pi x/l}$ whose Fourier coefficients $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ have finite h^α-norm.

2. L_q associated to T_r

For any given $r \in L_0^2(\mathbb{T})$ denote by T_r the impedance operator

$$T_r(u) \overset{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru' \tag{1}$$

on $L^2(\mathbb{T}_2)$ with domain $\mathrm{Dom}(T_r) = H^2(\mathbb{T}_2)$. Here ρ is the absolutely continuous, 1-periodic, positive function given by

$$\rho(x) \overset{\text{def}}{=} \exp\left(\int_0^x r(s)\,ds\right).$$

In particular, $\rho \in H^1(\mathbb{T})$ and $\rho' = r\rho$. Note that T_r is an operator with compact resolvent, nonnegative, and symmetric with respect to the inner product

$$(f, g)_\rho \overset{\text{def}}{=} \int_0^2 fg\rho^2\,dx \quad \text{on } L^2(\mathbb{T}_2)$$

(cf. Appendix A). Hence the spectrum $\mathrm{spec}(T_r)$ is discrete, real, and nonnegative, and the corresponding eigenvalues have the same (finite) algebraic and geometric multiplicities. It turns out that $\mathrm{spec}(T_r)$ is of the form

$$\mathrm{spec}(T_r) = \left\{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leqslant \tilde{\lambda}_2(r) \leqslant \cdots\right\}$$

(listed with multiplicities) and $\tilde{\lambda}_k(r) \to \infty$ as $k \to \infty$ (see Appendix A).

For any $q \in H^{-1}(\mathbb{T})$ we denote by L_q the Hill operator

$$L_q \overset{\text{def}}{=} -\frac{d^2}{dx^2} + q \tag{2}$$

viewed as an operator on the space $H^{-1}(\mathbb{T}_2)$ with domain $\mathrm{Dom}(L_q) = H^1(\mathbb{T}_2)$. The classical spectral theory of Hill's operator can be extended for such singular potentials (see Appendix B). The spectrum of L_q is discrete, real, and of the form

$$\mathrm{spec}(L_q) = \left\{\lambda_0(q) \leqslant \lambda_1(q) \leqslant \lambda_2(q) \leqslant \cdots\right\}$$

(see Lemma B.2, Appendix B). For each eigenvalue $\lambda_k(q)$, its algebraic multiplicity coincides with the geometric one. Further $\lambda_k(q) \to \infty$ as $k \to \infty$. Denote the eigenspace corresponding to an eigenvalue λ of L_q by

$$V_\lambda(L_q) \overset{\text{def}}{=} \left\{f \in \mathrm{Dom}(L_q) \;\middle|\; L_q f = \lambda f\right\}.$$

Similarly, we define $V_\lambda(T_r)$.

Clearly, for any $r \in L_0^2(\mathbb{T})$, $r^2 - \|r\|^2$ defines a bounded linear functional on $H^1(\mathbb{T})$ satisfying $\langle r^2 - \|r\|^2, 1\rangle = 0$ where $\|r\|^2 \overset{\text{def}}{=} \int_0^1 r^2(x)\,dx$. Hence, $r^2 - \|r\|^2$ is an element in $H_0^{-1}(\mathbb{T})$ and one can introduce the nonlinear map

$$R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T}), \quad r \mapsto r' + r^2 - \|r\|^2, \tag{3}$$

referred to as the Riccati map. The following result shows how the operators T_r and L_q are related if $q = R(r)$.

Lemma 1. *Let $r \in L_0^2(\mathbb{T})$ and assume that $q \in H_0^{-1}(\mathbb{T})$ satisfies Riccati's equation $q = R(r)$. Then*

(a) $\operatorname{spec}(T_r) = \|r\|^2 + \operatorname{spec}(L_q)$;
(b) *for any $k \geqslant 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\lambda_k + \|r\|^2}(T_r)$ have the same dimension*;
(c) *the $L^2(\mathbb{T})$-norm of r coincides with the absolute value of the first eigenvalue $\lambda_0(q)$ of Hill's operator L_q, i.e., $\|r\|^2 = |\lambda_0(q)| = -\lambda_0(q)$;*
(d) *the first eigenvalue $\lambda_0(q)$ of the operator L_q is simple and the corresponding eigenfunction normalized by $\|f_0\|^2 = 1$ and $f_0(0) > 0$ is $f_0 \stackrel{\text{def}}{=} \rho/\|\rho\|$. Hence, f_0 is in $H^1(\mathbb{T})$, does not vanish on $\mathbb{T}$, and satisfies $r = f_0'/f_0$.*

Proof. Let $f_k \in H^1(\mathbb{T}_2)$ be an eigenfunction of L_q with eigenvalue λ_k. Consider the function

$$\tilde{f}_k \stackrel{\text{def}}{=} \Phi_\rho(f_k) \stackrel{\text{def}}{=} f_k/\rho \in H^1(\mathbb{T}_2).$$

Note that T_r extends in a natural way to a continuous operator $\tilde{T}_r : H^1(\mathbb{T}_2) \to H^{-1}(\mathbb{T}_2)$.

By Lemma 2 below, Riccati's map takes the form $(\rho''/\rho) - \|r\|^2 = q$ where $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(s)\,ds)$. In this way we obtain

$$\begin{aligned}\tilde{T}_r(\tilde{f}_k) &\stackrel{\text{def}}{=} -\left(\rho^2 \frac{f_k'}{\rho} - f_k\rho'\right)' \Big/ \rho^2 = -\big(\rho f_k'' - f_k\rho''\big)/\rho^2 = \left(-f_k'' + \frac{\rho''}{\rho} f_k\right) \Big/ \rho \\ &= \big(\lambda_k + \|r\|^2\big)\tilde{f}_k. \end{aligned} \tag{4}$$

It follows from the above formulas that

$$\tilde{T}_r(\tilde{f}_k) \stackrel{\text{def}}{=} -\big(\rho^2 \tilde{f}_k'\big)' \big/ \rho^2 \in H^1(\mathbb{T}_2).$$

Using that ρ^2 is in $H^1(\mathbb{T}_2)$ and $H^1(\mathbb{T}_2)$ is an algebra, we obtain that $\rho^2 \tilde{f}_k' \in H^2(\mathbb{T}_2)$ and hence $\tilde{f}_k' \in H^1(\mathbb{T}_2)$. Therefore, we have proved that $\tilde{f}_k$ is in $H^2(\mathbb{T}_2)$ and thus an eigenfunction of T_r with eigenvalue $\tilde{\lambda}_k \stackrel{\text{def}}{=} \lambda_k + \|r\|^2$.

Conversely, let $\tilde{f}_k \in H^2(\mathbb{T}_2)$ be an eigenfunction of the impedance operator T_r with eigenvalue $\tilde{\lambda}_k$. Then $f_k \stackrel{\text{def}}{=} \Phi_\rho^{-1}(\tilde{f}_k) \stackrel{\text{def}}{=} \rho \tilde{f}_k$ is in $H^1(\mathbb{T}_2)$. Using Riccati's equation $(\rho''/\rho) - \|r\|^2 = q$ and Lemma 2 below, we obtain

$$\begin{aligned}L_q(f_k) &= -\rho''\tilde{f}_k - 2\rho'\tilde{f}_k' - \rho\tilde{f}_k'' + q\rho\tilde{f}_k = \rho\left(\left(-\tilde{f}_k'' - 2\frac{\rho'}{\rho}\tilde{f}_k'\right) - \|r\|^2\tilde{f}_k\right) \\ &= \big(\tilde{\lambda}_k(r) - \|r\|^2\big) f_k.\end{aligned}$$

Therefore, $\lambda_k \stackrel{\text{def}}{=} \tilde{\lambda}_k - \|r\|^2$ is an eigenvalue of Hill's operator L_q and item (a) of Lemma 1 is proved.

Item (b) follows from the previous arguments together with the invertibility of the map $\Phi_\rho : H^1(\mathbb{T}_2) \to H^1(\mathbb{T}_2)$ defined as multiplication by $1/\rho$.

Item (c) follows from item (a) and the fact that $\tilde{\lambda}_0(r) = 0$ is the first eigenvalue of the impedance operator T_r (see Appendix A).

To prove item (d), note that $\tilde{\lambda}_0(r) = 0$ is a simple eigenvalue of T_r and $\tilde{f}_0 \equiv 1/\|\rho\|$ a corresponding eigenfunction—see Appendix A for a proof. By item (b), $\lambda_0(q)$ is simple as well. The corresponding eigenspace is spanned by the function

$$f_0 \stackrel{\text{def}}{=} \Phi_\rho^{-1}\big(1/\|\rho\|\big) = \rho/\|\rho\|,$$

which is in $H^1(\mathbb{T})$ and does not vanish on $\mathbb{T}$. In particular, $f_0'/f_0 = \rho'/\rho$ and as ρ satisfies $r = \rho'/\rho$, it follows that $r = f_0'/f_0$, which finishes the proof of (d). $\square$

Remark 1. We point out that the operators $(\tilde{T}_r - \|r\|^2)$ and L_q are conjugated (by the map Φ_ρ) but the operators $(T_r - \|r\|^2)$ and L_q we consider in Lemma 1 are *not* conjugated.

Lemma 1(d) implies the following result:

Corollary 1. *The Riccati map $R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T})$ is injective. Moreover, if $q \in$ range(R), then $R^{-1}(q) = f_0'(\cdot,q)/f_0(\cdot,q)$ where $f_0(\cdot,q)$ is an eigenfunction corresponding to the lowest eigenvalue $\lambda_0(q)$ of L_q.*

Remark 2. The injectivity of R is due to Korotyaev [17, Theorem 2.5].

Remark 3. Note that the quotient f_0'/f_0 is independent of the normalization of f_0.

In the proof of Lemma 1 we have used the following lemma which can be proved in a straightforward way.

Lemma 2. *Let $u \in L^2(\mathbb{T}_2)$ and $v \in H^1(\mathbb{T}_2)$. Then the following statements hold*:

(a) *$uv \in L^2(\mathbb{T}_2)$ and $(uv)' = uv' + u'v$;*
(b) *if $v(x) \neq 0$ for every $x \in \mathbb{T}_2$ then $1/v \in H^1(\mathbb{T}_2)$ and $(1/v)' = -v'/v^2$.*

3. Riccati map

By definition, the Riccati map $R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T})$ is given by formula (3). For any $\alpha \geqslant 0$ denote by R_α the restriction of R to $H_0^\alpha(\mathbb{T}) \subseteq L_0^2(\mathbb{T})$, $R_\alpha \stackrel{\text{def}}{=} R|_{H_0^\alpha(\mathbb{T})}$.

Proposition 1. *For any $\alpha \geqslant 0$, the Riccati map $R_\alpha : H_0^\alpha(\mathbb{T}) \to H_0^{\alpha-1}(\mathbb{T})$ is a diffeomorphism from $H_0^\alpha(\mathbb{T})$ to $H_0^{\alpha-1}(\mathbb{T})$.*

Proof. Let us consider first the case $\alpha = 0$. By definition, $R_0 = R$. First note that R is continuous. The claimed statement then follows from the following three assertions:

(i) R has dense image;
(ii) R is surjective (and hence bijective by Corollary 1);

(iii) R and R^{-1} are differentiable.

Let

$$q \in C_0^\infty(\mathbb{T}) \stackrel{\text{def}}{=} \left\{ f \in C^\infty(\mathbb{T}) \,\middle|\, \int_0^1 f(x)\,dx = 0 \right\}$$

and $\lambda_0(q)$ be the first eigenvalue of L_q. By the classical theory of Hill's equation, the corresponding eigenfunction f_0 does not have zeroes (see [24]). Hence, $L_q f_0 = \lambda_0(q) f_0$ can be rewritten as

$$q - \lambda_0(q) = \frac{f_0''}{f_0} = r' + r^2,$$

where $r \stackrel{\text{def}}{=} f_0'/f_0$. Integrating the last equality, one gets $\lambda_0(q) = -\|r\|^2$, and therefore, $q = r' + r^2 - \|r\|^2 = R(r)$. As $C_0^\infty(\mathbb{T})$-functions are dense in $H_0^{-1}(\mathbb{T})$, the image of the Riccati map is dense in $H_0^{-1}(\mathbb{T})$ proving item (i).

To prove item (ii) take an arbitrary $q \in H_0^{-1}(\mathbb{T})$. It follows from item (i) that there exists a sequence $\{q_k\}_{k=1}^\infty \subseteq \text{range}(R)$ such that $q_k \to q$ $(k \to \infty)$ in $H_0^{-1}(\mathbb{T})$. Consider the sequence $\{r_k\}_{k=1}^\infty \subseteq L_0^2(\mathbb{T})$ such that $q_k = R(r_k)$. Item (c) of Lemma 1 shows that $\|r_k\|^2 = |\lambda_0(q_k)|$ where $\lambda_0(q_k)$ is the first eigenvalue of the Hill operator $L_{q_k} = -d^2/dx^2 + q_k$. As a function of the potential $q \in H_0^{-1}(\mathbb{T})$ the first eigenvalue $\lambda_0(q)$ is continuous on $H_0^{-1}(\mathbb{T})$—see Appendix B, Lemma B.2. Therefore, $\|r_k\| \to \sqrt{|\lambda_0(q)|}$ as $k \to \infty$. Hence, the sequence $\{r_k\}_{k=1}^\infty$ is bounded in $L^2(\mathbb{T})$.

Consider the map $S_0 : L^2(\mathbb{T}) \to H^{-1}(\mathbb{T})$, $r \mapsto r^2$. This map can be viewed as a composition of two maps $S_0 = \imath \circ S_1$, where $S_1 : L^2(\mathbb{T}) \to H^{-3/4}(\mathbb{T})$ is given by the formula $r \mapsto r^2$, and $\imath : H^{-3/4}(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$ is the standard inclusion of Sobolev spaces. By Rellich's theorem $\imath$ is compact. As each Fourier coefficient $\widehat{r^2}_n$ of r^2 satisfies $|\widehat{r^2}_n| \leqslant \|r\|^2$ $(n \in \mathbb{Z})$ it follows that there exists $C > 0$ so that $\|r^2\|_{-3/4} \leqslant C\|r\|^2$, hence the map S_1 is bounded. Therefore, there exists a subsequence $\{r_{k_j}\}_{j=1}^\infty$ of $\{r_k\}$ and an element $g \in H^{-1}(\mathbb{T})$ such that $r_{k_j}^2 \to g$ $(j \to \infty)$ in $H^{-1}(\mathbb{T})$. By the definition of Riccati's map,

$$r_{k_j}' = q_{k_j} - r_{k_j}^2 + \|r_{k_j}\|^2 \in H_0^{-1}(\mathbb{T}).$$

Each of the terms on the right-hand side of the latter equation converges in $H^{-1}(\mathbb{T})$. Hence r_{k_j}' converges to some element $s \in H_0^{-1}(\mathbb{T})$. Denote by r the unique element in $L_0^2(\mathbb{T})$ such that $s = r'$. As

$$\|r - r_{k_j}\| \leqslant \text{const} \big\| r' - r_{k_j}' \big\|_{-1} \to 0 \quad \text{for } j \to \infty$$

and R is continuous, it then follows that

$$q_{k_j} = R(r_{k_j}) \to R(r) \quad (j \to \infty) \quad \text{in } H^{-1}(\mathbb{T}).$$

On the other side, it follows from our construction that the sequence $\{q_{k_j}\}$ converges to q in $H^{-1}(\mathbb{T})$. Therefore, $q = R(r)$ and claim (ii) is proved.

Towards claim (iii) note that R is continuously differentiable. By (ii) and Corollary 1, $R^{-1}(q)$ is defined for any $q \in H_0^{-1}(\mathbb{T})$ and $R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q)$. By Lemma 1(d) for any $q \in H_0^{-1}(\mathbb{T})$, the first eigenvalue $\lambda_0(q)$ of L_q is simple and the corresponding eigenfunction $f_0(\cdot, q)$, normalized by $\int_0^1 f_0(x, q)^2\, dx = 1$ and $f_0(0, q) > 0$, is 1-periodic, does not vanish, and is an element in $H^1(\mathbb{T})$. By Theorem B.1 (Appendix B), $f_0(\cdot, q)$ considered as a map $H_0^{-1}(\mathbb{T}) \to H^1(\mathbb{T})$ is continuously differentiable (in fact real-analytic) in the variable q. This shows that R^{-1} is continuously differentiable.

In order to prove the theorem for arbitrary $\alpha \geqslant 0$, first note that by Lemma C.1(ii) (Appendix C), $R_\alpha(H^\alpha(\mathbb{T})) \subseteq H^{\alpha-1}(\mathbb{T})$. To prove that R_α is onto take an arbitrary $q \in H_0^{\alpha-1}(\mathbb{T})$. By the case $\alpha = 0$, $q = R(r)$ for some $r \in L_0^2(\mathbb{T})$. Denote by f_0 the eigenfunction corresponding to the first eigenvalue of the Hill operator L_q on $H^1(\mathbb{T}_2)$ normalized so that $\int_0^1 f_0(x, q)^2\, dx = 1$ and $f_0(0, q) > 0$. By item (d) of Lemma 1, f_0 is an element of $H^1(\mathbb{T})$, does not vanish on $\mathbb{T}$, and $r = f_0'/f_0$. Moreover, if $0 \leqslant \alpha \leqslant 1$ one concludes from $-f_0'' + qf_0 = \lambda_0 f_0$ and the fact that the product qf_0 belongs to $H^{\alpha-1}(\mathbb{T}_2)$ (see Lemma C.1(iii), Appendix C) that $f_0 \in H^{\alpha+1}(\mathbb{T}_2)$. If $\alpha > 1$, one argues similarly. Using Lemma C.1(iii) and (iv), one proves by induction that $f_0 \in H^{\alpha+1}(\mathbb{T}_2)$. As $f_0(x+1) = f_0(x)$, it follows that $f_0 \in H^{\alpha+1}(\mathbb{T})$ and as $H^{\alpha+1}(\mathbb{T})$ is an algebra $1/f_0 \in H^{\alpha+1}(\mathbb{T})$ and $f_0' \in H^\alpha(\mathbb{T})$. Therefore, $r = f_0'/f_0 \in H_0^\alpha(\mathbb{T})$. The claim that $R_\alpha^{-1}(q) = f_0'(q)/f_0(q)$ is continuously differentiable then follows from the same arguments as in the proof of (iii) for the case $\alpha = 0$—see Appendix B, Remark B.4. □

The following statement is a generalization of the corresponding classical result. Attempts to prove it using the classical approach (see [24]) fail at several stages.

Corollary 2. *For any $q \in H_0^{-1}(\mathbb{T})$, the first eigenvalue $\lambda_0(q)$ of L_q is simple. Any eigenfunction corresponding to $\lambda_0(q)$ is an element in $H^1(\mathbb{T})$ and does not vanish on $\mathbb{T}$. Moreover, $\|R^{-1}(q)\|$ is a spectral invariant of L_q.*

Proof. Take $r = R^{-1}(q)$. By Lemma 1(d) the first eigenvalue $\lambda_0(q)$ of L_q is simple and $f_0(x) = \rho(x)/\|\rho\|$ is an eigenfunction corresponding to $\lambda_0(q)$ where $\rho(x) = \exp(\int_0^x r(s)\, ds)$. Hence f_0 is an element in $H^1(\mathbb{T})$ and does not vanish on $\mathbb{T}$. It follows from Lemma 1(c) that $\|R^{-1}(q)\|^2 = -\lambda_0(q)$ which is obviously a spectral invariant. □

Denote by $H_0^\alpha(\mathbb{T}, \mathbb{C})$ the complexification of the (real) Sobolev space $H_0^\alpha(\mathbb{T})$. For complex-valued functions $r \in H_0^\alpha(\mathbb{T}, \mathbb{C})$, $\alpha \geqslant 0$, the (complex) Riccati map is defined by the formula

$$R_\alpha(r) \stackrel{\text{def}}{=} r' + r^2 - \int_0^1 r^2(x)\, dx.$$

Using the same arguments as in the real case, one concludes that R_α maps $H_0^\alpha(\mathbb{T}, \mathbb{C})$ into $H_0^{\alpha-1}(\mathbb{T}, \mathbb{C})$ and is an analytic map. As a consequence of Proposition 1 one easily obtains

Theorem 3. *For any $\alpha \geqslant 0$ there exist open neighborhoods $U \subseteq H_0^\alpha(\mathbb{T}, \mathbb{C})$ and $W \subseteq H_0^{\alpha-1}(\mathbb{T}, \mathbb{C})$ of $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ respectively such that the Riccati map $R_\alpha : U \to W$ is an analytic isomorphism.*

Proof. Consider first the case $\alpha = 0$. As we already mentioned above, the Riccati map

$$R : L_0^2(\mathbb{T}, \mathbb{C}) \to H_0^{-1}(\mathbb{T}, \mathbb{C}), \qquad R(r) \stackrel{\text{def}}{=} r' + r^2 - \int_0^1 r^2(x)\, dx,$$

is analytic.

It follows from Proposition 1 and Lemma 1(d) that for any $q \in H_0^{-1}(\mathbb{T})$ the first eigenvalue $\lambda_0(q)$ of L_q is simple and the corresponding (normalized) eigenfunction $f_0(\cdot, q)$ is 1-periodic, does not vanish, and $f_0 \in H^1(\mathbb{T})$. By Theorem B.1 (Appendix B) for any $q \in H_0^{-1}(\mathbb{T})$ there exists a (complex) neighborhood $W(q) \subseteq H_0^{-1}(\mathbb{T}, \mathbb{C})$ such that for any $p \in W(q)$ the eigenvalue $\lambda_0(p)$ is simple, the corresponding (normalized) eigenfunction $f_0(\cdot, p) \in H^1(\mathbb{T}, \mathbb{C})$ does not vanish and so that $f_0(\cdot, p)$ depends analytically on $p \in W(q)$. Moreover, we can chose $W(q)$ so small that

$$\operatorname{Re}\big(\lambda_0(p)\big) < \operatorname{Re}\big(\lambda_1(p)\big) \leqslant \cdots \quad \forall p \in W(q).$$

Consider the open (complex) neighborhood $W \stackrel{\text{def}}{=} \bigcup_{q \in H_0^{-1}(\mathbb{T})} W(q)$ of $H_0^{-1}(\mathbb{T})$ and define the map

$$B(q) = f_0'(\cdot, q)/f_0(\cdot, q), \qquad B : W \to U,$$

where $U \stackrel{\text{def}}{=} B(W) \subseteq L_0^2(\mathbb{T}, \mathbb{C})$. Clearly $B : W \to U$ is analytic.

By a simple verification one sees that $\forall q \in W$,

$$\begin{aligned} R\big(B(q)\big) &= \big(f_0'/f_0\big)' + \big(f_0'/f_0\big)^2 - \big\| f_0'/f_0 \big\|^2 = f_0''/f_0 - \big\| f_0'/f_0 \big\|^2 \\ &= q - \lambda_0(q) - \big\| f_0'/f_0 \big\|^2 = q \end{aligned}$$

and similarly for any $r \in U$, $B(R(r)) = r$. □

Proof of Theorem 1. The claimed results follow from Theorem 3 and Lemma 1. □

4. Applications

This section contains several applications of the Riccati representation of the elements of the Sobolev space $H_0^{-1}(\mathbb{T})$: we give asymptotic formulas for the spectrum of the Hill operator with a singular potential $q \in H^{-1}(\mathbb{T})$ and an a priori estimate of the potential q in terms of the gap lengths. Analogous asymptotic formulas are proved for the Dirichlet spectrum. Clearly, classical Floquet theory for Hill's operator $-d^2/dx^2 + q$ for potentials q in $L^2(\mathbb{T})$ cannot be extended to potentials in $H^{-1}(\mathbb{T})$. Nevertheless, we show that some features of the Floquet theory of Hill's operator can be extended without essential changes to the case of singular potentials from the Sobolev spaces $H^{-\alpha}(\mathbb{T})$, $0 < \alpha \leqslant 1$.

4.1. Periodic spectrum

In this paragraph we prove four results on the spectrum of Hill's operator. The first result can be deduced from results of Krein [19] and Theorem 1.

Theorem 4. *The spectrum of Hill's operator $L_q = -d^2/dx^2 + q$ on $H^{-1}(\mathbb{T}_2)$ with singular potential $q \in H^{-1}(\mathbb{T})$ is discrete,* $\mathrm{spec}(L_q) = \{\lambda_0(q) < \lambda_1(q) \leqslant \lambda_2(q) < \cdots\}$, *$\lambda_k(q) \to \infty$ as $k \to \infty$. The eigenvalues are totally ordered, $\lambda_{2k-1}(q) \leqslant \lambda_{2k}(q)$ and $\lambda_{2k}(q) < \lambda_{2k+1}(q)$, where the equality $\lambda_{2k-1}(q) = \lambda_{2k}(q)$ means that the corresponding eigenspace has two dimensions. Otherwise, the corresponding eigenspaces are one-dimensional. The eigenvalues $\lambda_{2k-1}(q) \leqslant \lambda_{2k}(q)$ with k odd correspond to anti-periodic eigenfunctions, i.e., $f(x+1) = -f(x)$ and those with k even to periodic ones, i.e., $f(x+1) = f(x)$.*

Proof. By Theorem 1, there exists $r \in L_0^2(\mathbb{T})$ such that $q = R(r)$. Theorem 4 then follows from Lemma 1 and the spectral properties of the impedance operator T_r (see Appendix A). □

For any $k \geqslant 0$, denote by

$$\gamma_k(q) \stackrel{\mathrm{def}}{=} \lambda_{2k}(q) - \lambda_{2k-1}(q)$$

the *kth gap-length* and by $\gamma(q)$ the sequence $\{\gamma_k(q)\}_{k\geqslant 1}$. The following two theorems are applications of results of Korotyaev [15] concerning the spectrum of the impedance operator T_r for $r \in L_0^2(\mathbb{T})$.

Theorem 5. *For any $q \in H^{-1}(\mathbb{T})$, $\{\gamma_k(q)\}_{k\geqslant 1}$ belongs to the sequence space h^{-1}. There exists a constant $c > 0$ such that for every potential $q \in H_0^{-1}(\mathbb{T})$,*

$$\|q\|_{-1} \leqslant c\big\|\gamma(q)\big\|_{-1}\big(1 + c\big\|\gamma(q)\big\|_{-1}\big)^3. \tag{5}$$

Proof. Without loss of generality we can assume that $q \in H_0^{-1}(\mathbb{T})$. Take $r \stackrel{\mathrm{def}}{=} R^{-1}(q)$. By Lemma 1, the operators L_q and T_r have, up to a translation, the same spectrum. Hence these operators have the same gap-lengths. The first statement in Theorem 5 thus follows from [15, Theorem 1.1].

Using

$$q = R(r) \stackrel{\mathrm{def}}{=} r' + r^2 - \|r\|^2,$$

the Cauchy–Schwartz inequality, and the easily verified inequalities $\|r'\|_{-1} \leqslant \|r\|$ and $\|r^2\|_{-1} \leqslant c_1\|r\|^2$ for some constant $c_1 > 0$, one concludes that there is a constant $c_2 > 0$ so that for any $r \in L_0^2(\mathbb{T})$, and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$,

$$\|q\|_{-1} = \big\|R(r)\big\|_{-1} \leqslant \|r'\|_{-1} + \big\|r^2\big\|_{-1} + \|r\|^2 \leqslant c_2\|r\|\big(1 + c_2\|r\|\big). \tag{6}$$

By [15, Theorem 1.2] for the impedance operator T_r, there exists $c_3 > 0$ so that for any

$r \in L_0^2(\mathbb{T})$ and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$,

$$\|r\| \leqslant c_3 \|\gamma(q)\|_{-1}\bigl(1 + c_3\|\gamma(q)\|_{-1}\bigr). \tag{7}$$

Combining these last two estimates, the last statement of Theorem 5 follows. $\square$

Remark 4. The first statement of Theorem 5 has also been obtained by Savchuk and Shkalikov [32]. Their proof does not use the spectral theory of the impedance operator.

Remark 5. By Lemma 1(d) and (7), we obtain the following estimate of the first eigenvalue $\lambda_0(q)$ in the terms of the sequence of gap lengths:

$$\sqrt{|\lambda_0(q)|} \leqslant c_3 \|\gamma(q)\|_{-1}\bigl(1 + c_3\|\gamma(q)\|_{-1}\bigr).$$

4.2. Dirichlet spectrum

Consider the operator $L_q^{\mathrm{Dir}} = -d^2/dx^2 + q$ on $H^{-1}[0,1] = (H_c^1[0,1])'$ with $q \in H^{-1}(\mathbb{T})$ and domain $\mathrm{Dom}(L_q^{\mathrm{Dir}}) = H^1_{\mathrm{Dir}}[0,1]$ (see Appendix B). First, we need some auxiliary results which again can be proved in a straightforward way.

Lemma 3. *For given $q \in H_0^{-1}(\mathbb{T})$, let $r \stackrel{\mathrm{def}}{=} R^{-1}(q) \in L_0^2(\mathbb{T})$ where R^{-1} is the inverse of Riccati's map. Then*

(a) $\mathrm{spec}(L_q^{\mathrm{Dir}}) = \mathrm{spec}(T_r^{\mathrm{Dir}}) - \|r\|^2$;
(b) *for any $k \geqslant 1$, the eigenspaces $V_{\mu_k}(L_q^{\mathrm{Dir}})$ and $V_{\mu_k+\|r\|^2}(T_r^{\mathrm{Dir}})$ both have dimension* 1.

Proof. The proof is almost the same as the one of Lemma 1. The only difference is that all calculations must be performed in the Sobolev space $H^{-1}[0,1]$ instead of $H^{-1}(\mathbb{T})$. This is possible using Lemma 4 below. $\square$

The following result is a version of Lemma 2 and can be proved in the same way.

Lemma 4. *For any $u \in L^2[0,1]$ and $v \in H^1[0,1]$ the following statements hold*:

(a) $uv \in L^2[0,1]$ *and* $(uv)' = uv' + u'v$;
(b) *if $v(x) \neq 0$ for every $x \in [0,1]$, then $1/v \in H^1[0,1]$ and $(1/v)' = -v'/v^2$.*

The next two theorems generalize results of [31].

Theorem 6. *The spectrum of L_q^{Dir} is discrete* $\mathrm{spec}(L_q^{\mathrm{Dir}}) = \{\infty < \mu_1(q) < \mu_2(q) < \cdots\}$, *the corresponding eigenspaces are one-dimensional, and $\mu_k \to \infty$ as $k \to \infty$.*

Proof. The claimed results follow from the spectral properties of the operator T_r^{Dir} (see Appendix A) and Lemma 3. $\square$

Theorem 7. *A sequence $-\infty < \sigma_1 < \sigma_2 < \cdots$ with $\sigma_k \to \infty$ as $k \to \infty$, is the Dirichlet spectrum of L_q^{Dir} for some $q \in H^{-1}(\mathbb{T})$, if and only if $\sigma_k = \mathrm{const} + (k\pi + s_k)^2$ where $\{s_k\}_{k\geqslant 1} \in h^0$.*

Remark 6. The (much easier) “only if” part of Theorem 7 is proved in [33].

Before proving Theorem 7 let us state Theorem 8. In [4,5] a real-analytic isomorphism has been constructed:

$$I \overset{\mathrm{def}}{=} \tilde{\mu} \times \tilde{\kappa} : L^2(\mathbb{T}) \to S \times h^0 \times \mathbb{R}, \quad r \mapsto \big(\tilde{\mu}(r), \tilde{\kappa}(r), -[r]\big).$$

The sequences $\tilde{\mu}(r)$ and $\tilde{\kappa}(r)$ in the formula above are defined as follows: $\tilde{\mu}(r) \overset{\mathrm{def}}{=} \{(\log \tilde{\mu}_k(r))/2\}_{k\geqslant 1}$ where $0 < \tilde{\mu}_1(r) < \tilde{\mu}_1(r) < \cdots$ is the spectrum of the impedance operator T_r^{Dir} and $\tilde{\kappa}(r) \overset{\mathrm{def}}{=} \{\tilde{\kappa}_k(r)\}_{k\geqslant 1}$, with $\tilde{\kappa}_k(r) \overset{\mathrm{def}}{=} \log\big|\frac{\tilde{g}_k'(1)}{\tilde{g}_k'(0)}\big|$ $(k \geqslant 1)$. Here $\tilde{g}_k \in H^2[0,1]$ is an eigenfunction of T_r^{Dir} corresponding to the eigenvalue $\tilde{\mu}_k$. The set S is defined by

$$S \overset{\mathrm{def}}{=} \big\{\big\{\log(k\pi + s_k)\big\}_{k\geqslant 1} \;\big|\; \{s_k\}_{k\geqslant 1} \in h^0,\ 0 < k\pi + s_k < (k+1)\pi + s_{k+1}\big\}.$$

Clearly, S can be identified with an open subset in h^0 and it is supplied with the topology and the (real) analytic structure induced from h^0. Composing R^{-1} with $I|_{L_0^2(\mathbb{T})}$, we obtain the mapping

$$J \overset{\mathrm{def}}{=} I|_{L_0^2(\mathbb{T})} \circ R^{-1} : H_0^{-1}(\mathbb{T}) \to S \times h^0, \quad q \mapsto \big(\mu(q), \kappa(q)\big),$$

which is a real-analytic isomorphism as well. Note that for $q \in H_0^{-1}(\mathbb{R})$, the function $r \overset{\mathrm{def}}{=} R^{-1}(q)$ has mean zero, $[r] = 0$. Let us express $\mu(q)$ and $\kappa(q)$ as functions of the potential $q \in H_0^{-1}(\mathbb{T})$. Given $q \in H_0^{-1}(\mathbb{T})$ let $r \overset{\mathrm{def}}{=} R^{-1}(q)$. The eigenvalues $\tilde{\mu}_k$ of T_r^{Dir} satisfy $0 < \tilde{\mu}_1 < \tilde{\mu}_2 < \cdots$ and, according to Lemma 3, $\tilde{\mu}_k = \mu_k + \|R^{-1}(q)\|^2$ where $\{\mu_k\}_{k\geqslant 1}$ is the spectrum of L_q^{Dir}. Then $\mu(q)$ is given by

$$\mu(q) = \big\{\log\big(\big(\mu_k + \big\|R^{-1}(q)\big\|^2\big)^{1/2}\big)\big\}_{k\geqslant 1}.$$

The sequence $\kappa(q)$ is given by $\{\kappa_k(q)\}_{k\geqslant 1}$ where $\kappa_k(q) = \log\big|\frac{\tilde{g}_k'(1)}{\tilde{g}_k'(0)}\big|$. Unlike an eigenfunction $\tilde{g}_k$ of the operator T_r^{Dir} an eigenfunction of L_q^{Dir} corresponding to an eigenvalue $\mu_k = \tilde{\mu}_k - \|R^{-1}(q)\|^2$ is not necessarily in $H^2[0,1]$ but only in $H^1[0,1]$ in the case where $q \in H_0^{-1}(\mathbb{T})$. However, if $q \in L_0^2(\mathbb{T})$, any such eigenfunction g_k is in $H^2[0,1]$ and we have $\frac{g_k'(1)}{g_k'(0)} = \frac{\tilde{g}_k'(1)}{\tilde{g}_k'(0)}$.

Theorem 8. *The mapping $q \mapsto (\mu(q), \kappa(q))$ is a real-analytic isomorphism onto $S \times h^0$.*

Proof of Theorems 7 and 8. The stated results follow directly from [5, Corollaries 5.5, 5.6] together with Lemma 3 and Theorem 3. $\square$

4.3. Discriminant of Hill's operator

The potentials $q \in H_0^{-1}(\mathbb{T})$ are too singular for L_q to admit fundamental solutions. Hence the Floquet matrix cannot be defined in this situation. However, it turns out that the trace $\Delta(\lambda, q)$ of the Floquet matrix, referred to as discriminant, can still be defined as we will explain now. Recall that the discriminant $\tilde{\Delta}$ of the impedance operator T_r is defined for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T})$ arbitrary, by $\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r)$ where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation

$$-u'' - 2ru' = \tilde{\lambda}u. \tag{8}$$

For $q \in L_0^2(\mathbb{T})$ the discriminant $\Delta(\lambda, q)$ is well-defined and related to $\tilde{\Delta}(\tilde{\lambda}, r)$ as follows. Define $r \stackrel{\text{def}}{=} R^{-1}(q)$ and $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(s)\,ds)$. As $q \in L^2(\mathbb{T})$, the equation

$$-y'' + qy = \lambda y \tag{9}$$

admits fundamental solutions $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$, i.e., solutions satisfying the initial conditions $y_1(0, \lambda, q) = y_2'(0, \lambda, q) = 1$ and $y_1'(0, \lambda, q) = y_2(0, \lambda, q) = 0$. It is easy to see that for $j = 1, 2$, the functions $\tilde{y}_j(x, \tilde{\lambda}, r) \stackrel{\text{def}}{=} y_j(x, \lambda, q)/\rho(x)$ are solutions of (8) with $\tilde{\lambda} = \lambda + \|r\|^2$ and $\|r\|^2 = \int_0^1 r^2(x)\,dx$. As $\rho(0) = 1$, one obtains $\tilde{y}_1(0, \tilde{\lambda}, r) = \tilde{y}_2'(0, \tilde{\lambda}, r) = 1$, $\tilde{y}_2(0, \tilde{\lambda}, r) = 0$, and $\tilde{y}_1'(0, \tilde{\lambda}, q) = -\rho'(0)$. Hence, the fundamental solutions u_1 and u_2 of Eq. (8) are related to $\tilde{y}_1$ and $\tilde{y}_2$ by

$$u_1(x, \tilde{\lambda}, r) = \tilde{y}_1(x, \tilde{\lambda}, r) + \rho'(0)\tilde{y}_2(x, \tilde{\lambda}, r)$$

and

$$u_2(x, \tilde{\lambda}, r) = \tilde{y}_2(x, \tilde{\lambda}, r).$$

Using that $\rho(1) = \rho(0) = 1$ and $\rho'(0) = \rho'(1)$, we obtain

$$\begin{aligned}
\tilde{\Delta}(\tilde{\lambda}, r) &\stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r) \\
&= \big(y_1/\rho + \rho'(0)y_2/\rho\big)\big|_{(x=1,\lambda,q)} + \big(y_2'/\rho - y_2\rho'/\rho^2\big)\big|_{(x=1,\lambda,q)} \\
&= y_1(1, \lambda, q) + y_2'(1, \lambda, q) \stackrel{\text{def}}{=} \Delta(\lambda, q).
\end{aligned} \tag{10}$$

Hence, we can *define* $\Delta(\lambda, q)$ for $q \in H_0^{-1}(\mathbb{T})$ by the latter identity.

Definition 1. For any $q \in H_0^{-1}(\mathbb{T})$ and $\lambda \in \mathbb{C}$,

$$\Delta(\lambda, q) \stackrel{\text{def}}{=} \tilde{\Delta}\big(\lambda + \|r\|^2, r\big), \tag{11}$$

where $r = R^{-1}(q)$, $\|r\|^2 = \int_0^1 r^2(x)\,dx$.

It follows directly from Theorem 3 and the properties of $\tilde{\Delta}(\tilde{\lambda}, r)$ (cf. [4, Lemmas 1.1 and 1.2]) that $\Delta(\lambda, q)$ is an analytic function on $\mathbb{C} \times W$ where $W \subseteq H_0^{-1}(\mathbb{T}, \mathbb{C})$ is the neighborhood of $H_0^{-1}(\mathbb{T})$ given by Theorem 3, and that the zeroes of $\Delta(\lambda, q)^2 - 4$ are precisely the eigenvalues of L_q, i.e., for any $\lambda \in \mathbb{C}$,

$$\Delta(\lambda, q)^2 - 4 = 0 \quad \text{if and only if} \quad \lambda \in \operatorname{spec}(L_q).$$

4.4. Isospectral invariance of Riccati's map

For any $q \in H_0^{-1}(\mathbb{T})$ denote by $\mathrm{Iso}(L_q)$ the set of potentials $p \in H_0^{-1}(\mathbb{T})$ such that $\mathrm{spec}(L_p) = \mathrm{spec}(L_q)$, i.e.,

$$\mathrm{Iso}(L_q) \stackrel{\text{def}}{=} \bigl\{p \in H_0^{-1}(\mathbb{T}) \bigm| \mathrm{spec}(L_p) = \mathrm{spec}(L_q)\bigr\}.$$

Similarly for any $r \in L_0^2(\mathbb{T})$, denote

$$\mathrm{Iso}(T_r) \stackrel{\text{def}}{=} \bigl\{u \in L_0^2(\mathbb{T}) \bigm| \mathrm{spec}(T_u) = \mathrm{spec}(T_r)\bigr\}.$$

Theorem 9. *For every $r \in L_0^2(\mathbb{T})$,*

$$R\bigl(\mathrm{Iso}(T_r)\bigr) = \mathrm{Iso}(L_{R(r)}).$$

Proof. Let $r \in L_0^2(\mathbb{T})$ and $q \stackrel{\text{def}}{=} R(r) \in H_0^{-1}(\mathbb{T})$. To see that $\mathrm{Iso}(L_{R(r)}) \subseteq R(\mathrm{Iso}(T_r))$ take any $p \in \mathrm{Iso}(L_q)$ and set $u \stackrel{\text{def}}{=} R^{-1}(p)$. By Lemma 1(a) and (d),

$$\mathrm{spec}(T_u) = -\lambda_0(p) + \mathrm{spec}(L_p) = -\lambda_0(q) + \mathrm{spec}(L_q) = \mathrm{spec}(T_r).$$

Conversely, take $u \in \mathrm{Iso}(T_r)$ and let $p \stackrel{\text{def}}{=} R(u)$. By the definition of the isospectral set $\mathrm{Iso}(T_r)$, we obtain that $\mathrm{spec}(T_u) = \mathrm{spec}(T_r)$. It follows from Lemma 1(a) that $\mathrm{spec}(L_p) = \mathrm{spec}(T_u) - \|u\|^2$ and $\mathrm{spec}(L_q) = \mathrm{spec}(T_r) - \|r\|^2$. By Corollary A.1 (Appendix A), the L^2-norm $\|r\|$ of the potential $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r. Hence, it follows from $\mathrm{spec}(T_u) = \mathrm{spec}(T_r)$ that $\|u\|^2 = \|r\|^2$ and therefore $\mathrm{spec}(L_p) = \mathrm{spec}(L_q)$. This completes the proof of Theorem 9. □

Corollary 3. *For every potential $q \in H_0^{-1}(\mathbb{T})$, the isospectral set* $\mathrm{Iso}(L_q)$ *is compact in $H_0^{-1}(\mathbb{T})$.*

Proof. First we prove that for any $r \in L_0^2(\mathbb{T})$ the isospectral set $\mathrm{Iso}(T_r)$ is compact. Let $\{r_k\}_{k\geqslant 0}$ be a sequence in $\mathrm{Iso}(T_r)$. It follows from the spectral invariance of the L^2-norm of r that $\|r_k\| = \|r\|$ for any $k \geqslant 1$ (Corollary A.1, Appendix A). Hence, there exists a subsequence $\{r_{k_j}\}_{j\geqslant 1}$ that converges weakly to some element $u \in L_0^2(\mathbb{T})$. The sequence of discriminants $\tilde{\Delta}(\tilde{\lambda}, r_k)$ of T_{r_k} converges to $\tilde{\Delta}(\tilde{\lambda}, u)$ as $k \to \infty$ uniformly on bounded subsets of $\mathbb{C}$ (cf. [4, Lemmas 3.1 and 3.2]). On the other side, the spectral invariance of r_k shows that $\tilde{\Delta}(\tilde{\lambda}, r_k) = \tilde{\Delta}(\tilde{\lambda}, r)$ and therefore $\tilde{\Delta}(\tilde{\lambda}, r) = \tilde{\Delta}(\tilde{\lambda}, u)$. In particular, $\mathrm{spec}(T_r) = \mathrm{spec}(T_u)$ and $\|u\| = \|r\|$. As r_{k_j} converges weakly to u, $\|u - r_{k_j}\|^2 = 2\|u\|^2 - 2(u, r_{k_j}) \to 0$ as $j \to \infty$. Therefore, the isospectral set $\mathrm{Iso}(T_r)$ is compact.

Theorem 9 and the continuity of the Riccati map $R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T})$ then imply the compactness of the isospectral sets $\mathrm{Iso}(L_q)$, $q \in H_0^{-1}(\mathbb{T})$. □

4.5. Complex potentials

In a straightforward way many of the previous results can be extended for complex potentials in some open neighborhood W of $H_0^{-1}(\mathbb{T})$ in $H_0^{-1}(\mathbb{T}, \mathbb{C})$. As an example we mention the following theorem which can be proved using the same arguments as in the proof of Lemma 1.

Denote by U and W the neighborhoods given by Theorem 3.

Theorem 10. *For given $q \in W \subseteq H_0^{-1}(\mathbb{T},\mathbb{C})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in U \subseteq L_0^2(\mathbb{T},\mathbb{C})$ where R^{-1} is the inverse of the Riccati map $R : U \to W$. Then*

$$\operatorname{spec}(L_q) = \operatorname{spec}(T_r) - \int_0^1 r(x)^2\,dx.$$

The same relation is true for the Dirichlet spectra of these operators.

In view of Theorem 10 one can reformulate results on the spectrum of the impedance operator T_r with $r \in U$ in terms of the corresponding result for the operator L_q with $q = R(r) \in W$, and vice versa.

Appendix A. Impedance operator

An impedance operator is a Sturm–Liouville operator of a special type and is treated in numerous articles and books—see [4,5,15,19,20,22]. For the convenience of the reader we recall its properties needed in the main part of this paper.

A.1. Periodic problem

For any $r \in L_0^2(\mathbb{T})$ denote by ρ the element in $H^1(\mathbb{T})$ satisfying $\rho' = r\rho$ and $\rho(0) = 1$. Then ρ is a one-periodic, absolutely continuous, positive function given by $\rho(x) = \exp(\int_0^x r(s)\,ds)$. The periodic impedance operator T_r is defined on the Hilbert space $L^2(\mathbb{T}_2)$ with domain $\operatorname{Dom}(T_r) = H^2(\mathbb{T}_2)$ by the formula

$$T_r(u) \stackrel{\text{def}}{=} -\big(\rho^2 u'\big)'/\rho^2 = -u'' - 2ru'. \tag{A.1}$$

Note that the operator T_r is positive and symmetric with respect to the $L^2(\mathbb{T}_2)$-inner product $(f,g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2\,dx$.

The impedance operator T_r has compact resolvent. Indeed, it is clear that $T_r|_{H^2(\mathbb{T}_2)}$ considered as a map $H^2(\mathbb{T}_2) \to L^2(\mathbb{T}_2)$ is continuous. Let us choose $\mu \in \mathbb{C}$ such that $\tilde{\Delta}(\mu)^2 \neq 4$ (cf. below for the definition of the discriminant $\tilde{\Delta}(\lambda)$ of the impedance operator T_r and its basic properties) and consider the differential equation

$$-u'' - 2ru' - \mu u = f, \tag{A.2}$$

where $f \in L^2(\mathbb{T}_2)$. The general solution of (A.2) can be written (uniquely) in the form $u(x) = c_1u_1(x) + c_2u_2(x) + v(x)$, $c_1, c_2 = \text{const}$, where $u_1(x)$ and $u_2(x)$ are the fundamental solutions of the homogeneous equation $-u'' - 2ru' - \mu u = 0$ and $v(x)$ is a particular solution of (A.2) with initial data $v|_{x=0} = 0$ and $v'|_{x=0} = 0$.[2] As $\tilde{\Delta}(\mu)^2 \neq 4$, one easily sees that the constants c_1 and c_2 can be chosen (uniquely) so that $u|_{x=0} = u|_{x=2}$ and $u'|_{x=0} = u'|_{x=2}$, and therefore $u \in H^2(\mathbb{T}_2)$. Hence,

$$T_r|_{H^2(\mathbb{T}_2)} - \mu : H^2(\mathbb{T}_2) \to L^2(\mathbb{T}_2)$$

is a continuous bijection and by the open mapping theorem

$$(T_r|_{H^2(\mathbb{T}_2)} - \mu)^{-1} : L^2(\mathbb{T}_2) \to H^2(\mathbb{T}_2)$$

is continuous.[3] As the inclusion $H^2(\mathbb{T}_2) \to L^2(\mathbb{T}_2)$ is compact, the resolvent $(T_r - \mu)^{-1} : L^2(\mathbb{T}_2) \to L^2(\mathbb{T}_2)$ is compact.

As T_r has compact resolvent, the spectrum of T_r is discrete and as this operator is symmetric with respect to the above mentioned inner product, its eigenvalues have the same (finite) algebraic and geometric multiplicities. It turns out (see [19,22]) that $\operatorname{spec}(T_r)$ is of the form $\operatorname{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leqslant \tilde{\lambda}_2(r) \leqslant \cdots\}$, the corresponding eigenspaces are of dimension 1 or 2, and $\tilde{\lambda}_k(r) \to \infty$ as $k \to \infty$. For $k \geqslant 0$ even, the eigenfunctions $\tilde{f}_{2k-1}$ and $\tilde{f}_{2k}$ are periodic while for k odd, $\tilde{f}_{2k-1}$ and $\tilde{f}_{2k}$ are anti-periodic. The eigenvalues $\lambda_{2k-1}(r)$ and $\lambda_{2k}(r)$ coincide iff the corresponding eigenspaces have dimension 2. All these properties follow from results in [19] as the impedance operator T_r can be transformed by the change of the variable $y = y(x) \stackrel{\text{def}}{=} \int_0^x (1/\rho^2(s))\, ds$ to the linear operator $-\rho^{-4}\frac{d^2}{dy^2}$ on the torus $\mathbb{T}_l$ with period $l \stackrel{\text{def}}{=} y(1)$ whose spectral properties are established in [19].

Lemma A.1. *The first eigenvalue $\tilde{\lambda}_0(r) = 0$ of T_r is simple and the corresponding eigenspace is spanned by the constant function $\tilde{f}_0 = 1/\|\rho\|$.*

Proof. Assume that $u \in \ker(T_r)$. Integrating by parts, we obtain

$$0 = \big(T_r(u), u\big)_\rho \stackrel{\text{def}}{=} -\int_0^2 \big(\rho^2 u'\big)' u\, dx = \int_0^2 \rho^2 (u')^2\, dx. \tag{A.3}$$

As ρ is positive, $u' \equiv 0$ and hence u is constant. This shows that the dimension of $\ker(T_r)$ is equal to one and Lemma A.1 is proved. □

The discriminant $\tilde{\Delta}$ of the impedance operator T_r is defined for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T})$ arbitrary, by

$$\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r),$$

where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation $-u'' - 2ru' = \tilde{\lambda} u$. It follows from the results in [4] (cf. Lemmas 1.1 and 1.2) that the discriminant

[2] Recall (cf. [19]) that a $C^1([0, 2])$ function u is a solution of (A.2) iff u' is an absolutely continuous function on $[0, 2]$ such that $u'' \in L^1([0, 2])$ and (A.2) is satisfied a.e. on $[0, 2]$. It is not hard to see that any solution of (A.2) indeed belongs to $H^2([0, 2])$.

[3] In particular, the graph of T_r is closed.

$\tilde{\Delta}(\tilde{\lambda}, r)$ is an analytic function on $\mathbb{C} \times L_0^2(\mathbb{T})$. The following lemma can be proved in straightforward way using the results in [4,19].

Lemma A.2. *The discriminant $\tilde{\Delta}(\tilde{\lambda}, r)$ is a spectral invariant of the impedance operator T_r. The set of zeroes $\tilde{\lambda}_k$ of the equation $\tilde{\Delta}(\tilde{\lambda}, r)^2 = 4$, counted with their multiplicities, coincides with the spectrum of T_r.*

Lemma A.2 together with [16, Corollary 1.2] then lead to the following corollary.

Corollary A.1. *The L^2-norm $\|r\|$ of $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r.*

A.2. Dirichlet problem

The Dirichlet problem for the impedance operator has been considered by many authors—see, e.g., [4,5,22]. The operator T_r^{Dir} is defined on $L^2[0,1]$ with domain

$$\mathrm{Dom}\big(T_r^{\mathrm{Dir}}\big) = H^2_{\mathrm{Dir}}[0,1] \stackrel{\mathrm{def}}{=} \big\{ f \in H^2[0,1] \bigm| f(0) = f(1) = 0 \big\}.$$

By definition, T_r^{Dir} acts on elements $u \in H^2_{\mathrm{Dir}}[0,1]$ by the formula

$$T_r^{\mathrm{Dir}}(u) \stackrel{\mathrm{def}}{=} -\big(\rho^2 u'\big)'/\rho^2 = -u'' - 2ru',$$

where

$$\rho(x) \stackrel{\mathrm{def}}{=} \exp\left(\int_0^x r(s)\, ds \right).$$

The spectrum $\mathrm{spec}(T_r^{\mathrm{Dir}})$ is called *Dirichlet spectrum* of T_r. It is known that $\mathrm{spec}(T_r^{\mathrm{Dir}})$ is discrete, all eigenvalues are simple, and $\mathrm{spec}(T_r^{\mathrm{Dir}}) = \{0 < \mu_1(r) < \mu_2(r) < \cdots\}$ where $\mu_k(r) \to \infty$ as $k \to \infty$. Other spectral properties of T_r^{Dir}, including the solution of an inverse problem, were established in [4,5].

Appendix B. Schrödinger operator

B.1. Periodic problem

Take $q \in H^{-1}(\mathbb{T}) \stackrel{\mathrm{def}}{=} (H^1(\mathbb{T}))'$ and consider Hill's operator

$$L_q = -\frac{d^2}{dx^2} + q \tag{B.1}$$

on $H^{-1}(\mathbb{T}_2)$ with domain $\mathrm{Dom}(L_q) = H^1(\mathbb{T}_2)$. The elements of $H^{-1}(\mathbb{T})$ can be considered as elements of $H^{-1}(\mathbb{T}_2)$ as follows: to any element $u \in H^{-1}(\mathbb{T})$ with Fourier expansion $u = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{i2k\pi x}$, we assign the unique element in $H^{-1}(\mathbb{T}_2)$ given by the

Fourier series $\sum_{k\in\mathbb{Z}} \hat{s}_k e^{ik\pi x}$ with $\hat{s}_{2k} \stackrel{\text{def}}{=} \hat{u}_k$ and $\hat{s}_{2k+1} \stackrel{\text{def}}{=} 0$. The operator L_q acts on elements $u \in H^1(\mathbb{T}_2)$ by $L_q u \stackrel{\text{def}}{=} -u'' + qu$ where the multiplication qu is viewed as an element of $H^{-1}(\mathbb{T}_2)$ according to the formula $\langle qu, v\rangle \stackrel{\text{def}}{=} \langle q, uv\rangle$. The brackets $\langle\cdot,\cdot\rangle$ denote the dual pairing between the elements of $H^{-1}(\mathbb{T}_2) \stackrel{\text{def}}{=} (H^1(\mathbb{T}_2))'$ and $H^1(\mathbb{T}_2)$. As the multiplication map $H^1(\mathbb{T}_2) \times H^1(\mathbb{T}_2) \to H^1(\mathbb{T}_2)$ given by $u \cdot v \stackrel{\text{def}}{=} uv$ is continuous, the linear functional qu is continuous as well. It can be easily seen that L_q induces a bounded operator $L_q : H^1(\mathbb{T}_2) \to H^{-1}(\mathbb{T}_2)$. Considered as an operator on $H^{-1}(\mathbb{T}_2)$, with domain $\mathrm{Dom}(L_q) = H^1(\mathbb{T}_2)$, L_q is an unbounded operator.[4]

The operators L_q with singular potentials $q \in H^{-\alpha}(\mathbb{T})$, $0 < \alpha \leqslant 1$, have been considered in [7,29] (cf. [30,32] for a different approach). In order to make this paper self-contained, we review the case $\alpha = 1$ treated in [29] and give the auxiliary facts used in the main part of the paper. In our presentation, we mainly follow [29, §1.5.1] (cf. [7]).

For $M > 0$, $r > 0$, and $n \in \mathbb{N}$ introduce the sets

$$\mathrm{Ext}_M \stackrel{\text{def}}{=} \bigl\{\lambda \in \mathbb{C} \bigm| \mathrm{Re}(\lambda) \leqslant \bigl|\mathrm{Im}(\lambda)\bigr| - M\bigr\} \tag{B.2}$$

$$\mathrm{Vert}_n(r) \stackrel{\text{def}}{=} \bigl\{\lambda = n^2\pi^2 + z \in \mathbb{C} \bigm| \bigl|\mathrm{Re}(z)\bigr| \leqslant n\pi^2,\ |z| \geqslant r\bigr\}. \tag{B.3}$$

Via the Fourier transform, we identify L_q with the operator $\hat{L}_v$ on h^{-1} with domain $\mathrm{Dom}(\hat{L}_v) = h^1$. The operator $\hat{L}_v$ acts on the sequences $x = \{x_k\}_{k\in\mathbb{Z}} \in h^1$ by $D + V$ where D and V are the infinite matrices $D \stackrel{\text{def}}{=} (k^2\pi^2\delta_{kl})_{k,l\in\mathbb{Z}}$ and $V \stackrel{\text{def}}{=} (v(k-l))_{k,l\in\mathbb{Z}}$, δ_{kl} is the Kronecker delta and $v(k)$ are the Fourier coefficients of the potential $q \in H^{-1}(\mathbb{T})$ viewed as an element in $H^{-1}(\mathbb{T}_2)$. The proof of the following auxiliary result can be found in [13, Appendix B] (cf. also [29]).

Lemma B.1. *For every $v \in h^{-1}$ there exist a neighborhood $U(v) \subseteq h^{-1}$ of v and constants $M > 0$ and $n_0 \in \mathbb{N}$ such that for every $u \in U(v)$, the sets Ext_M and $\mathrm{Vert}_n(n)$ $(n > n_0)$ are contained in the resolvent set $\mathrm{resol}(\hat{L}_u)$ of $\hat{L}_u$. The resolvent $(\lambda - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$, considered as a function of (λ, u) on $\mathrm{Ext}_M \times U(v)$ or $\mathrm{Vert}_n(n) \times U(v)$ with $n > n_0$, is continuous in (λ, u) and for every $u \in U(v)$ holomorphic in λ. Moreover, for any smooth contour $\Gamma \subseteq \mathrm{Ext}_M \cup \bigcup_{n>n_0} \mathrm{Vert}_n(n)$ and integer $l \geqslant 0$, the operator*

$$Q_\Gamma^l(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_\Gamma \lambda^l (\lambda - \hat{L}_u)^{-1}\, d\lambda \in \mathcal{L}\bigl(h^{-1}, h^1\bigr)$$

is analytic when viewed as a map $U(v) \to \mathcal{L}(h^{-1}, h^1)$.

Using Lemma B.1 and arguing as in [29], one proves the following results.

[4] For complex valued potentials $q \in H^{-1}(\mathbb{T}, \mathbb{C})$ one defines L_q in a similar way as an unbounded operator on $H^{-1}(\mathbb{T}_2, \mathbb{C})$.

Lemma B.2.

(i) *For any* $q \in H^{-1}(\mathbb{T})$, L_q *has a compact resolvent.*
(ii) *The spectrum of the Hill operator* L_q *with potential* $q \in H^{-1}(\mathbb{T})$ *is discrete,* $\operatorname{spec}(L_q) = \{\lambda_0(q) \leqslant \lambda_1(q) \leqslant \lambda_2(q) \leqslant \cdots\}$, *and* $\lambda_k(q) \to \infty$ *as* $k \to \infty$. *For each eigenvalue* $\lambda_k(q)$, *its algebraic multiplicity coincides with the geometric one.*
(iii) *As functions of the potential* $q \in H^{-1}(\mathbb{T})$, *the kth eigenvalue* $\lambda_k(q) : H^{-1}(\mathbb{T}) \to \mathbb{R}$, $q \mapsto \lambda_k(q)$ *is continuous.*
(iv) *Suppose that the eigenvalue* $\lambda_k(q)$ *is simple*[5] *for some* $q \in H^{-1}(\mathbb{T})$. *Then there exists a neighborhood* $U(q) \subseteq H^{-1}(\mathbb{T})$ *of* q *such that for any* $u \in U(q)$, *the kth eigenvalue* $\lambda_k(u)$ *is simple and the corresponding eigenfunction* $f_k(\cdot, u)$ (*normalized so that* $\int_0^2 f_k^2(x, u)\, dx = 2$ *and* $f(x_0, u) > 0$, *where* $x_0 \in [0, 2]$ *is chosen so that* $f_k(x_0, q) > 0$ *for the given potential* q) *is analytic as a map* $U(q) \to H^1(\mathbb{T}_2)$.

Remark B.1. We improve on Lemma B.2 in Section 4 (see Theorem 4).

Remark B.2. An analogue of Lemma B.2 is true for potentials $q \in H^{\alpha-1}(\mathbb{T})$ with arbitrary $\alpha \geqslant 0$. In particular, the eigenvalues $\lambda_k(q)$ are continuous with respect to the norm in $H^{\alpha-1}(\mathbb{T})$. If $\lambda_k(q)$ is simple, the corresponding normalized eigenfunction $f_k(\cdot, q)$ is analytic as a map from a neighborhood of q in $H^{\alpha-1}(\mathbb{T})$ to $H^{\alpha+1}(\mathbb{T}_2)$.

Remark B.3. Note that $\hat{L}_v = D + V$ leaves invariant the subspaces $h_\pm^{-1}$ of h^{-1} where

$$h_+^{-1} \stackrel{\text{def}}{=} \left\{(a_k) \in h^{-1} \;\middle|\; a_{2k+1} = 0\ \forall k \in \mathbb{Z}\right\} \quad \text{and}$$
$$h_-^{-1} \stackrel{\text{def}}{=} \left\{(a_k) \in h^{-1} \;\middle|\; a_{2k} = 0\ \forall k \in \mathbb{Z}\right\}.$$

Lemma B.2 can be extended to complex potentials as follows.

Theorem B.1. *The spectrum* $\operatorname{spec}(L_q) = \{\lambda_k\}_{k \geqslant 0}$ *of Hill's operator* $L_q = -d^2/dx^2 + q$ *on* $H^{-1}(\mathbb{T}_2, \mathbb{C})$ *with singular potential* $q \in H^{-1}(\mathbb{T}, \mathbb{C})$ *is discrete, the eigenvalues being ordered lexicographically,* $\operatorname{Re}(\lambda_0(q)) \leqslant \operatorname{Re}(\lambda_1(q)) \leqslant \operatorname{Re}(\lambda_2(q)) \leqslant \cdots$. *The corresponding root spaces are of finite dimension, and* $\operatorname{Re}(\lambda_k(q)) \to \infty$ *as* $k \to \infty$. *If the eigenvalue* $\lambda_k(q)$, $q \in H^{-1}(\mathbb{T})$, *is simple and the corresponding* (*one-dimensional*) *eigenspace consists of* 1*-periodic functions then there exists a* (*complex*) *neighborhood* $U(q) \subseteq H^{-1}(\mathbb{T}, \mathbb{C})$ *such that for any* $p \in U(q)$ *the eigenvalue* $\lambda_k(p)$ *is simple and analytic considered as a function* $U(q) \to \mathbb{C}$. *The corresponding eigenfunction* $f_k(\cdot, p) \in H^1(\mathbb{T}, \mathbb{C})$ (*normalized by* $\int_0^1 f_k(x, p)^2\, dx = 1$ *and* $\operatorname{Re}(f_k(x_0, p)) > 0$ *for some given appropriately chosen* $x_0 \in \mathbb{T}$) *depends analytically on* $p \in U(q)$.

Remark B.4. The same theorem is true for potentials $q \in H^{-1+\alpha}(\mathbb{T}, \mathbb{C})$ for $\alpha \geqslant 0$.

[5] I.e., λ_k has algebraic multiplicity 1.

B.2. Dirichlet problem

The aim of this paragraph is to set up the Dirichlet problem for the operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ on $[0,1]$ with potential $q \in H^{-1}(\mathbb{T})$. In order to make our presentation self-contained, we give some definitions and auxiliary facts on this problem in the present section. Further details (including the case $q \in H^{-\alpha}(\mathbb{T})$, $\alpha < 1$) can be found in [7, §3].

Define the operator L_q^{Dir} on the Sobolev space $H^{-1}[0,1]$ with domain $\text{Dom}(L_q^{\text{Dir}}) = H^1_{\text{Dir}}[0,1]$. By definition,

$$H^1_{\text{Dir}}[0,1] \stackrel{\text{def}}{=} \{ f \in H^1[0,1] \mid f(0) = f(1) = 0 \} \quad \text{and}$$
$$H^{-1}[0,1] \stackrel{\text{def}}{=} \left(H^1_{\text{Dir}}[0,1] \right)'$$

—see Section 1 where the definition of the Sobolev space $H^1[0,1]$ is recalled. In a natural way, the elements of $H^1_{\text{Dir}}[0,1]$ can be identified with elements in $H^1(\mathbb{T})$, the corresponding inclusion map $\imath : H^1_{\text{Dir}}[0,1] \to H^1(\mathbb{T})$ being continuous. For any $u \in H^1_{\text{Dir}}[0,1]$, qu is defined to be the functional in $H^{-1}[0,1] \stackrel{\text{def}}{=} (H^1_{\text{Dir}}[0,1])'$ given by the formula

$$\langle qu, v \rangle \stackrel{\text{def}}{=} \langle q, \imath(u)\imath(v) \rangle.$$

As the multiplication map $H^1(\mathbb{T}) \times H^1(\mathbb{T}) \to H^1(\mathbb{T})$, $f \cdot g \stackrel{\text{def}}{=} fg$, is continuous, the mapping $H^{-1}[0,1] \times H^1_{\text{Dir}}[0,1] \to H^{-1}[0,1]$, $(q,u) \mapsto qu$, is continuous as is the operator $\frac{d^2}{dx^2} : H^1[0,1] \to H^{-1}[0,1]$. For any $u \in H^1_{\text{Dir}}[0,1]$ we set $L_q^{\text{Dir}} u \stackrel{\text{def}}{=} -\frac{d^2 u}{dx^2} + qu$. In this way, L_q^{Dir} is a bounded operator $L_q^{\text{Dir}} : H^1_{\text{Dir}}[0,1] \to H^{-1}[0,1]$. Considered as an operator on $H^{-1}[0,1]$, L_q^{Dir} is an unbounded operator.

Lemma B.3. *The operator L_q^{Dir} has compact resolvent. As a consequence, the spectrum of L_q^{Dir} is discrete, the eigenspaces are of finite dimension, and in every compact set $K \subseteq \mathbb{C}$ there are finitely many eigenvalues.*

Proof. As in [7, §3.2] we identify the operator L_q^{Dir} with an operator on an appropriate sequence space. Then Lemma B.3 can be proved using the same arguments as in the proof of Lemma B.2. □

Appendix C. Convolution lemma

The convolution product $a * b$ of two sequences

$$a \stackrel{\text{def}}{=} \{a(k)\}_{k \in \mathbb{Z}} \subseteq \mathbb{C} \quad \text{and} \quad b \stackrel{\text{def}}{=} \{b(k)\}_{k \in \mathbb{Z}} \subseteq \mathbb{C}$$

is formally defined as the sequence given by

$$(a * b)(k) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} a(k-j)b(j).$$

Lemma C.1.

(i) *For any* $l \in \mathbb{Z}$ *the convolution* $(a, b) \mapsto a * b$ *is continuous, uniformly in* l, *when viewed as a map* $h^1 \times h^{-1,l} \to h^{-1,l}$ *and* $h^{-1} \times h^{1,l} \to h^{-1,l}$.

(ii) *For any* $\alpha \geqslant 0$, *the convolution* $(a, b) \mapsto a * b$ *is continuous when viewed as a map* $h^\alpha \times h^\alpha \to h^{\alpha-1}$.

(iii) *For any* $0 \leqslant \alpha \leqslant 1$, *the convolution* $(a, b) \mapsto a * b$ *is continuous when viewed as a map* $h^1 \times h^{\alpha-1} \to h^{\alpha-1}$.

(iv) *For any* $\alpha \geqslant 0$ *and any* $\epsilon > 0$, *the convolution* $(a, b) \mapsto a * b$ *is continuous when viewed as a map* $h^\alpha \times h^{\alpha+1/2+\epsilon} \to h^\alpha$.

The proof of Lemma C.1 is straightforward.

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